SECONDARY TEACHERS' RELATIVE SIZE SCHEMES[1]

Cameron Byerley, Patrick W. Thompson

Arizona State University

This paper explores the usefulness of understanding quotients as measures of relative size in mathematics. The paper characterizes the types of thinking displayed by high school mathematics teachers on two novel tasks designed to reveal teachers' meanings in contexts where making comparisons of relative size is productive.

INTRODUCTION

Comparing the relative size of two quantities is an important mental operation that can be employed productively to reason about topics that span the grades 2-12 curriculum. These topics include measurement, fractions, rates, slope, trigonometry, and derivatives. After discussing topics where conceiving of the relative size of two quantities is useful, we will describe results from a study designed to reveal meanings held by 100 high school teachers in regard to two items where conceptions of relative size are useful.

Comparisons of relative size are critical in conceiving of a quantity's measure. The measure of some quantity tells us how many times as large the quantity is as the unit by which it is measured. Second grade students are supposed to measure an object using two different sized units and describe how measurements are related to the size of the unit chosen (CCSS.Math.Content.2.MD.A.1, 2010). Third grade students are asked to understand the quotient 32/8 as telling us that 32 is some number of times as large as 8 (CCSS.Math.Content.OA.B.6). This meaning for quotient helps students make sense of situations where division is used. For example, 25 inches divided by 12 inches per foot tells us that 25 inches is 25/12 times as large as the standard measure of one foot. Thus 25/12 feet is the same magnitude as 25 inches. Fourth grade students are asked to know the relative sizes of units within one measurement system (CCSS.Math.Content.2.MD.A.1) as well as express measurements given in a larger unit in terms of a smaller unit (CCSS.Math.Content.4.MD.A.2). Fifth grade students are asked to convert among different sized measurement units within a given measurement system (CCSS.Math.Content.5.MD.A.1).

Fractions are a critical part of the Common Core curriculum in grades three through five. Thompson and Saldanha (2003) discuss the utility of conceiving of fractions as reciprocal relationships of relative size. The fraction p/q tells us how many times as large p is as q. Reciprocally, q is q/p times as large as p. Middle school students continue to study multiplicative relationships between two covarying quantities. For example, a rate can be considered to be a measure of the relative sizes of changes in

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two quantities. Constant speed measured in miles per hour tells us that the number of miles traveled is so many times as large as the number of hours elapsed. Two quantities change proportionally if, as each quantity changes, the relative size of the two quantities' changes is constant. If an object is traveling at a constant speed, any change in the measure of distance is always the same number of times as large as the associated change in the measure of time (Thompson, 1994).

In algebra, conceiving of slope as a measure of how many times as large a change in y is as a change in x is useful in modelling and writing equations of lines. In trigonometry, radian measure can be thought of as a measure of how many times as large an arc length is as the radius of the given circle. In calculus, the difference quotient $(f(x+h) - f(x))/h$ can be understood as comparing the relative size of $f(x+h) - f(x)$ and h. We believe this meaning for difference quotient is more useful than thinking of "$f(x+h) - f(x)$ out of h" or "go up $f(x+h) - f(x)$ for every time we go over h" because these meanings do not work well when h is small. Additionally, because h becomes increasingly small and is typically not equal to one, thinking of slope as how much y increases for a one-unit change in x is not productive in calculus. This list represents only a subset of mathematical topics where considerations of relative size are productive.

THEORETICAL PERSPECTIVE: QUANTITATIVE REASONING AND MEANING

The theoretical perspective guiding the creation and scoring of the items reported evolved from the work of Piaget and von Glasersfeld. A project team designs items to reveal secondary teacher's mathematical meanings. The intent of our items is find out what meanings teachers have with regard to various mathematical concepts; notice, this is not equivalent to classifying teachers into categories according to those who can solve a problem and those who cannot. This theoretical perspective, and what we mean by "meaning" is addressed in depth in Thompson, et al.(2013). For any mathematical idea, there are a variety of potential meanings, some of which are more useful than others because of the coherence they provide a teachers' thinking and instruction. For example, the meaning of quotient as a measure of relative size would allow a teacher to explain why division is used in the slope formula.

It is possible to have multiple meanings for one topic, and each meaning can be either quantitative or computational. For instance, a computational meaning for quotient held by some calculus students is that quotient is the answer that results from performing long division (Byerley, Hatfield, & Thompson, 2012). We attempt to determine whether a teacher's meaning is computational or is based on reasoning about the quantities in the item.

Much has been written on student and teacher understandings of the curricular topics connected to conceptions of relative size such as fractions, rates of change and derivatives (Armstrong & Bezuk, 1995; Bowers & Doerr, 2001; Harel & Behr, 1995; Izsák, Jacobson, de Araujo, & Orrill, 2012; Orton, 1983; Steffe & Olive, 2010). See

Sowder et al. (1998) for a good overview of the literature related to teachers' understandings of multiplicative structures. In short, there is much evidence that both teachers and students struggle with topics that have a comparison of relative size at the heart of the idea.

METHODOLOGY

The two items discussed in this paper are part of the assessment project *Mathematical Meanings for Teaching secondary mathematics* (MMTsm). Items in the MMTsm were developed based on conceptual models of thinking that arose from prior research, our teaching or from interviewing teachers and students. For example, prior research on quotient (Ball, 1990; Ma, 1999; Simon, 1993) shows that both elementary and secondary mathematics teachers have stronger computational meanings than quantitative meanings. In items where teachers were asked to create a story problem for division by a fraction, most did not demonstrate a meaning for quotient as the relative size of two quantities. Coe (2007), Castillo-Garsow (2010) and Johnson (2010) found that often secondary students' and teachers' meanings for rate of change did not entail the idea of relative size of changes.

Items went through a process with multiple revisions as a result of doing item interviews with teachers, showing items to mathematicians and math educators, and analysing data from approximately 150 teachers collected in summer 2012. Further details of the methodology were described in a methodology paper submitted to PME 38 (P. W Thompson & Draney, under review).

RESULTS

In the results section we will present two items, the rationale behind the items, and the teachers' results on the items. The first item, shown in Figure 1, was created to reveal teacher's meanings for constant speed.

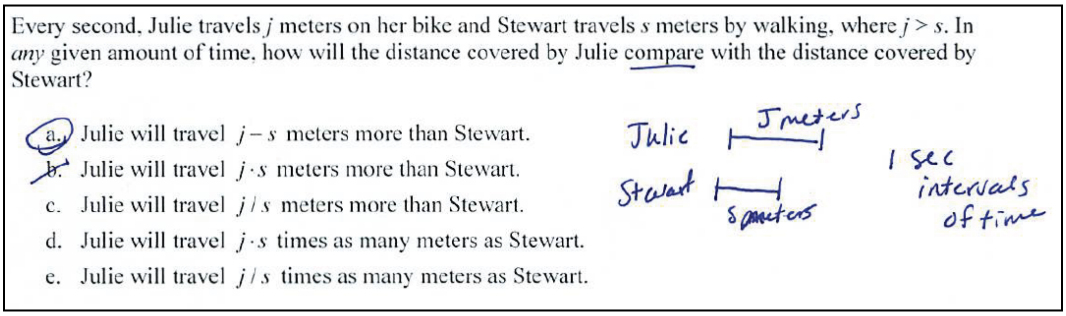
Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j-s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j/s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j/s times as many meters as Stewart.

Figure 1: An item on relative rates.

Based on prior research we hypothesized that some teachers' meanings for speed were "chunky" (Castillo-Garsow, 2010). For those with a chunky meaning, speed is the distance travelled in a 1-unit interval (i.e. chunk) of time as opposed to a measure of how many times as large the measure of distance travelled is as the measure of elapsed

time. We suspected that teachers with chunky meanings for speed might choose *j-s*, an answer that is only true for the first one-second interval. There is some evidence in the written work and interview data to support this hypothesis, an example of which is provided in Figure 2.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a. Julie will travel $j - s$ meters more than Stewart.
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c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

Figure 2: A teacher's response.

In teacher responses to other items, interviews, and in the literature, we also noticed teachers using the formula $d = rt$ inappropriately and thought that some teachers may expect to see a product as part of the answer (Bowers & Doerr, 2001). For example, some teachers used the formula $d = rt$ to find the total distance travelled on a trip with a non-constant rate of change by simply selecting the rate of change at the end of the trip. We have not yet done interviews to see why teachers selected $j*s$ in (b), (d). The only difference between response (c) and (e) is that the response (e) uses multiplicative language and (c) uses additive language. When scoring the items we do not think of teachers' answer in terms of correct and incorrect, but in terms of how productive those meanings are for teaching. In this case we believe (e) is the most productive way to think about speed because it generalizes to situations where the change in time is not one-unit. Some teachers who substituted values for j and s were able to determine that the quotient j/s was important but did not select the phrasing "times as many" and instead chose (c).

The teachers' responses to "Relative Rates" are shown in Table 1.

Response	Math Majors	Math Ed Majors	Other Majors	Total
j-s	15	22	10	47
$j*s$ more	0	0	4	4
j/s more	4	5	3	12
$j*s$ times	0	1	5	6
j/s times	5	10	15	30
"no time"	1	0	0	1
Total	25	38	37	100

Table 1: High school teacher's responses to "Relative Rates."

The majority of responses (70%) do not reflect multiplicative comparisons of the relative size of distance travelled for any amount of elapsed time. Teachers with "Other" degrees (e.g., Art History, Biology or Religion) were more likely to choose the highest-level response (40%) than teacher with math (20%) or math education degrees (26%), but we did not find this relationship to be statistically significant.

We designed the second item, shown in Figure 3, to see whether teachers' thinking about a relative size situation would be constrained by the quantitative relationships or would be primarily algorithmic. Although the item is most closely aligned with elementary measurement standards, this foundational understanding is important in secondary mathematics standards as well. The first quantitative relationship is that when the magnitude of the unit is increased, the measure of the container will decrease. The second relationship is that if the new unit is 189/50 times as large as the old unit, the measure of the container is 50/189 times as large in the new unit.

> A container has a volume of *m* liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 3: The second item, "Liters to Gallons."

We scored responses to Liters to Gallons from 100 high school math teachers using a rubric that was negotiated by the project team. Responses that omitted *m* or did not somehow indicate the idea of "number of liters" at level zero (e.g. the teacher only wrote *189/50*). Uninterpretable responses, responses that cubed part of the expression to find volume, and responses of "I don't know" were also scored at level zero. For example, Figure 4 shows a level zero response from a math major who has taught forty high school math courses.[2]

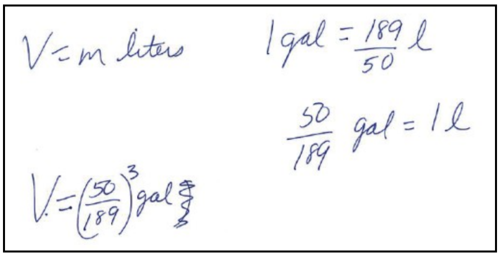

Figure 4: Level zero response to liters to gallons.

This was not the only response that used a cubic term in the answer and some stated explicitly that there must be three variables or that something must be cubed in volume problems. These responses do not reflect an awareness of how the relative size of the two units influences the relative size of the two measures.

The response in Figure 5 from a teacher with a math education degree who has taught seven high school math courses is an additional example of a level zero response. The teacher used the letter *G* to refer to both the magnitude of one gallon in the first line, and the number of gallons in the second line. The teacher did not demonstrate awareness of the reciprocal relationship between the measure of a quantity and the size of the unit measuring it.

[2] For example, if a teacher taught one Algebra class, four geometry classes, and one study skills class in a school year we would say they taught five math courses that year.

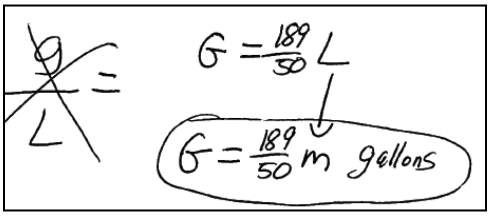

Figure 5: Example of level zero response to “Liters to Gallons”.

The response *G = 189/50 m* is level one if the teacher never used the same letter to represent two different quantities. The teacher in Figure 5 would have been scored at level one if the response had not used the letter “*G*” to represent two different quantities.

Level two responses demonstrate the correct relationship of relative size between the volume in gallons and the number of liters, by using the reciprocal 50/189. However they are not scored at the highest level because teachers wrote that a volume in gallons is a number of liters. If the response in Figure 6 had omitted the word “liters” in the final line or wrote that 50/189 had units of gallons per liter, the response would have been considered highest level.

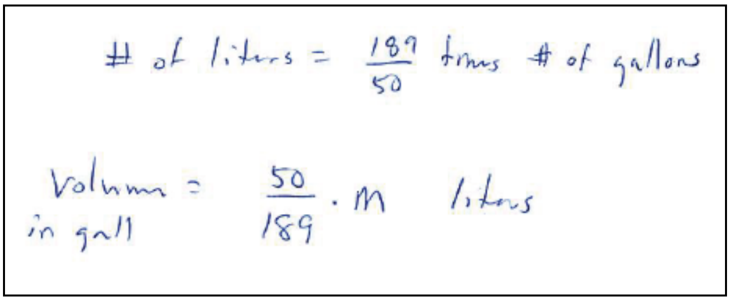

Figure 6: Example of a level two response.

Correct answers with explanations and correct answers without explanations were both scored at level three. Sometimes the response was only written symbolically such as *(50/189)m*. Level three responses may have incorrect work crossed out, but the teacher settled on a response of “the number of gallons equals *(50/189)* times *m*.”

Response	Math Majors	Math Ed Majors	Other Majors	Total
Level 0	5	8	12	25
189/50 m	14	20	14	48
50/189 m liters	1	1	1	3
50/189 m gallons	5	9	10	24
Total	25	38	37	100

Table 2: Responses and degree type for "Liters to Gallons."

The majority of responses (63%), regardless of teacher degree, demonstrate that the teacher did not consider the quantitative relationships regarding relative size when producing their answer. Although Table 2 shows that teachers with Other Majors have a higher percentage of highest-level responses (27%) than Math Majors (20%) and Math Ed Majors (23%), we found no statistically significant relationship between degree type and level of response.

The MMTsm had one additional item involving the comparison of two measures. The item in Figure 7 included an image of a circle with a highlighted arc.

In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?

Figure 7: Additional measurement item named "Nerds and Raps."

Most teachers answered either 9 or 16, with 50 out of 100 high school teachers giving a highest-level response of 16. Out of those 50 teachers who had a highest-level response to “Nerds and Raps” only 17 gave a highest-level response to “Liters and Gallons.” We hypothesize using the letter “*m*” to represent an arbitrary number of liters required additional meanings for variables or increased the likelihood of using algebraic computations without considering quantitative relationships of relative size. However, even if the difficulty of Gallons to Liters was primarily caused by the variable “*m*”, Nerds and Raps shows at least 50% of the teachers were not constrained by quantitative relationships of relative size-in this case, the smaller the unit, the larger the measure.

After meeting and interviewing a number of teachers who responded to these items, we suspect their daily work does not require them to consider quantitative relationships of relative size. We believe most teachers are capable of reasoning quantitatively, but they have had few occasions to do so. When we used the Gallons to Liters problem in a workshop for teachers who took the MMTsm, they were able to think about it quantitatively. We believe MMTsm can be used in professional development to help teachers develop more productive meanings. We have conducted one three-day professional development using a number of our items that was positively received by the teachers, but further research is needed to help teachers build meanings related to relative size.

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